A Framework for Research into Students' Problem Posing in School Mathematics

Elena Stoyanova and Nerida F. Ellerton
Edith Cowan University

Research into the potential of problem posing as a means for developing of students' understanding of mathematics has been hindered by the absence of a framework which links problem solving, problem posing and mathematics curricula. This paper presents an overview of the frameworks used by researchers for investigating problem posing, and proposes a framework for research into students' problem posing in mathematics. Examples of problem-posing situations used in a classroom with mathematically able students are presented.

Definitions of Problem Posing

The notion of problem posing has been explored by different researchers from contrasting perspectives. For example, problem posing has been viewed as the generation of a new problem or reformulation of a given problem (Duncer, 1945); as the formulation of a sequence of mathematical problems from a given situation (Shukkwan, 1993); or as a resultant activity when a problem is inviting the generation of other problems (Mamona-Downs, 1993). Dillon (1982) conceptualised "problem finding as a process resulting in a problem to solve."

Silver (1993, 1995) referred to problem posing as involving the creation of a new problem from a situation or experience, or the reformulation of given problems. Such problem posing could occur prior to problem solving (when problems are being generated from a given contrived or naturalistic situation), during the problem solving process (one can intentionally change some of the problem's goals or conditions), or after solving a particular problem (as would be the case when problems are generated on the basis of the experience gained by solving a particular problem or a set of problems).

In this paper mathematical problem posing will be defined *as the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems.*

The definition is deliberately broad to enable problem posing to fit within the goals of mathematical instruction in the context of school mathematics. The broaden definition also means that the researcher can explore the interrelationships between problem posing and problem solving as a means of mathematical instruction, and can examine the design of a wider range of problem-posing situations.

Recognition of Problem Posing

Many prominent scientists have recognised that the ability to *pose* significant questions had an equally important role to play in their scientific work as the ability to *solve* them. Einstein and Insfeld (1938), for example, wrote: "The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to regard old questions from a new angle, requires creative imagination and marks real advance in science"(p. 92).

The significance of the solution of a specific problem depends, to very large extent, on the significance of the question asked. In his investigation on "expert" and "novice" scientists, Zuckerman (1977) found that the "expert" scientists differed from the "novices" not so much in the *answers* as in the *questions* that the two groups of scientists posed.

In mathematics education, after over a decade of studies which have focused on problem solving, researchers have slowly begun to realise that developing the ability to *pose* mathematics problems is at least as important, educationally, as developing the ability to *solve* them. Commentators such as Resnick and Klopfer (1989) have recognised that helping students become competent thinkers is a central challenge for all educators. It has also been recognised that the incorporation of problem-posing activities into regular classroom situations can be a powerful approach for developing students' mathematical thinking (Silver, Kilpatrick & Schlesinger, 1990).

The mathematics curriculum documents of several countries have acknowledged the impact which problem posing could have on mathematical instruction. In Australia, for example, *The National Statement on Mathematics for Australian Schools* offered strong support for the use of open-ended problems in mathematics classrooms with the words: "Students should engage in extended mathematical activities which encourage problem posing, divergent thinking, reflection and persistence. They should be expected to pursue alternative strategies, and to pose and attempt to answer their own mathematical questions" (Australian Education Council, 1991, p. 39).

In the United States, *The Curriculum and Evaluation Standards for School Mathematics*, (National Council of Teachers of Mathematics, 1989) acknowledged the importance of having students experience some of the problem-posing aspects involved in the work of mathematicians: "Students in grade 9-12 should also have some experience recognising and formulating their own problems, an activity that is at the heart of doing mathematics" (p. 138). Investigative mathematical projects have been used in Victoria for assessing students' knowledge and skills since the late 1980s (Stacey, 1995).

There is a growing interest towards incorporating problem posing-activities into mathematics classroom (see, for example, Brown & Walter, 1983, 1990, 1993; Kilpatrick, 1987; Silver & Cai, 1993; Silver & Mamona, 1989; Silver, Kilpatrick & Schlesinger, 1990), and researchers have tried to use different frameworks for exploring problem posing. This movement makes it all the more important for researchers to develop appropriate frameworks for exploring problem posing.

Research Framework for Exploring Students' Problem Posing in Mathematics

Research into the potential of problem posing as an important strategy for the development of students' understanding of mathematics has been hindered by the absence of a framework which links problem solving, problem posing and mathematics curricula. Before the effects of problem posing and its implication for the teaching and learning of mathematics can be adequately researched, such a framework needs to be developed and refined in the light of data gained from its application in the classroom. This paper outlines the basis of such a framework, and emphasises the potential offered by extending Krutetskii's problem-solving categories as problem-posing situations.

Central to the framework proposed in this paper is the notion that every problem-posing situation can be classified as free, semi-structured or structured. All three categories have, in effect, been used by researchers for investigating various aspects of the effects of problem posing on mathematical instruction, but a framework which places a range of problem-posing situations into one of these three categories has not been proposed before.

We will describe a problem-posing situation as *free*, when students are asked to generate a problem from a given, contrived or naturalistic situation. Some directions may be given to prompt certain specific actions. Many researchers have used free problem-posing situations in their studies. For example, Ellerton (1986a, 1986b) introduced creative writing in mathematics by asking students to make up mathematics problems. She asked Australian students to pose a problem which would be difficult for a friend to solve. She also asked students to write a letter to a friend, who had been away ill from school, describing the mathematics which the class had done during the past 3 weeks. As part of the letter, students were asked to make up mathematics questions which were typical of those which they had encountered during the same period. She used this framework as window into exploring students' perceptions of mathematics. According to Ellerton (1988), "children's expression of mathematical ideas through the creation of their own mathematics problems demonstrates not only their understanding and level of concept development, but also reflects their perception about the nature of mathematics" (p. 281).

Richardson and Williamson (1982) used another form of free writing. They asked children to make up mathematical problems for each other. In his study, Kennedy (1985) used forms of writing to his mathematics students as writing letters about what were they studying, keeping logs and devising mathematical problems about a particular topic. Problem-posing activities involving much younger children have been described by Van der Brink (1985), who asked Grade 2 children to make up problems and games for Grade 1 children.

A problem-posing situation will be referred to as *semi-structured* when students are given an open situation and are invited to explore the structure and to complete it by applying knowledge, skills, concepts and relationships from their previous mathematical experiences. Hart (1981), for example, asked children to make up mathematics problems to fit given computations. Her aim was to study how children draw on concrete situations in describing symbolic expressions. Winograd (1991) used posing and sharing story problems as a research tool and found that children generally composed problems which they themselves had difficulty understanding or solving. Writz and Kahn (1982) observed that having students make up applications helped them to bridge the gap between concrete situations and mathematical abstractions. Furthermore, it appeared to help students to learn how to generalise, as well as making mathematics more meaningful to them. Students who wrote problem stories tended to learn to integrate mathematics with other subject areas and to develop creative writing skills (Bush & Fiala, 1986).

A problem posing situation will be called *structured* when problem-posing activities are based on a specific problem. In order to reveal the structure of students' mathematical abilities Krutetskii (1976) used a research tool involving students in finishing or reconstructing a specific problem structure. In his study he used problems with unstated questions, problems with insufficient and problems with surplus information.

In several studies, researchers have asked students to pose problems similar to a given problem as a tool for exploring some aspects of their mathematical performance. Hashimoto (1987), for example, found that asking students to pose a problem *similar* to a solved problem can be a useful teaching technique for providing a mirror into students' understanding of mathematical concepts.

Stover (1982) investigated the consequences of having students make format changes to mathematics problems. In this study, sixth-grade students were asked to modify one of three structural format variables (by adding a diagram, or removing extraneous information, or reordering information) in the statement of a problem, and observed substantial improvement in students' ability to solve word problems of the type they had learned to modify. Smilansky (1984) investigated the relationship between being able to solve problems and to pose problems in the same domain. After he had collected students' responses to a mathematical test, he distributed a skeleton test page and asked the pupils to create new problem which would be particularly difficult in the future version of the test. Smilansky found a low correlation between the performance on the problem-solving task and the problem-posing task in the same domain.

An Investigation of Problem Posing in the Classroom

In order to investigate the range of problem-posing situations which could be used as part of problem solving environments, a one-year study with mathematically able children was undertaken. A total of 40 Years 8 and 9 students from different schools in Perth, were involved in a mathematics enrichment program. Students took part in the program for one hour per week, from the beginning of February to mid-November, 1995.

The design of the problem posing situations was based on the following basic assumptions: (a) problem-posing situations should correspond to, and arise naturally out of, pupils' classroom mathematics activities; (b) problem-posing situations could be generated from textbook problems, by modifying and reshaping the language and task characteristics; (c) problem-posing situations should be a part of pupils' problem-solving activities.

Most of the structured problem-posing categories were inspired by Krutetskii's (1976) work. In fact, although Krutetskii's major focus was problem solving, his insights into the relationship between problem solving and problem posing has prompted us to reflect on how his ideas could be extended to embrace both problem solving and problem posing. This infers that Krutetskii's problem-solving categories can be readily applied by educators wishing to develop quality structured problem-posing situations for mathematics classrooms.

Free Problem-Posing Situations

In order to encourage students to reflect on their specific previous experience, the free problem-posing situations used in this study were addressed to problem posers, or placed the problem posers in a situation where they were forced to consider the person(s) for whom they were posing the problem. For example, students were asked to pose problems for mathematics competitions (see Figure 1); problems which they like (see Figure 2); problems which they found difficult (see Figure 3); and problems which would have to be solved by their teacher.

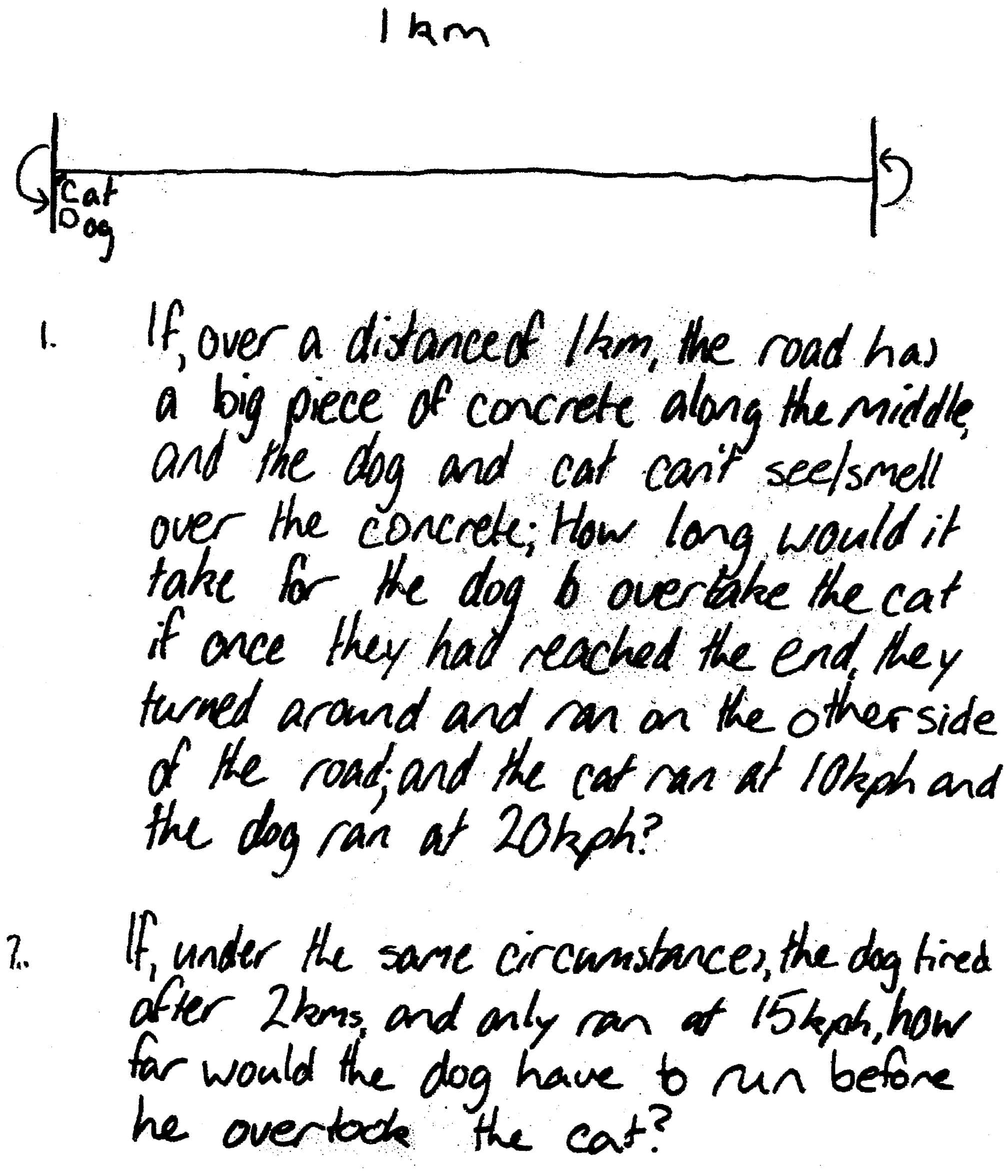
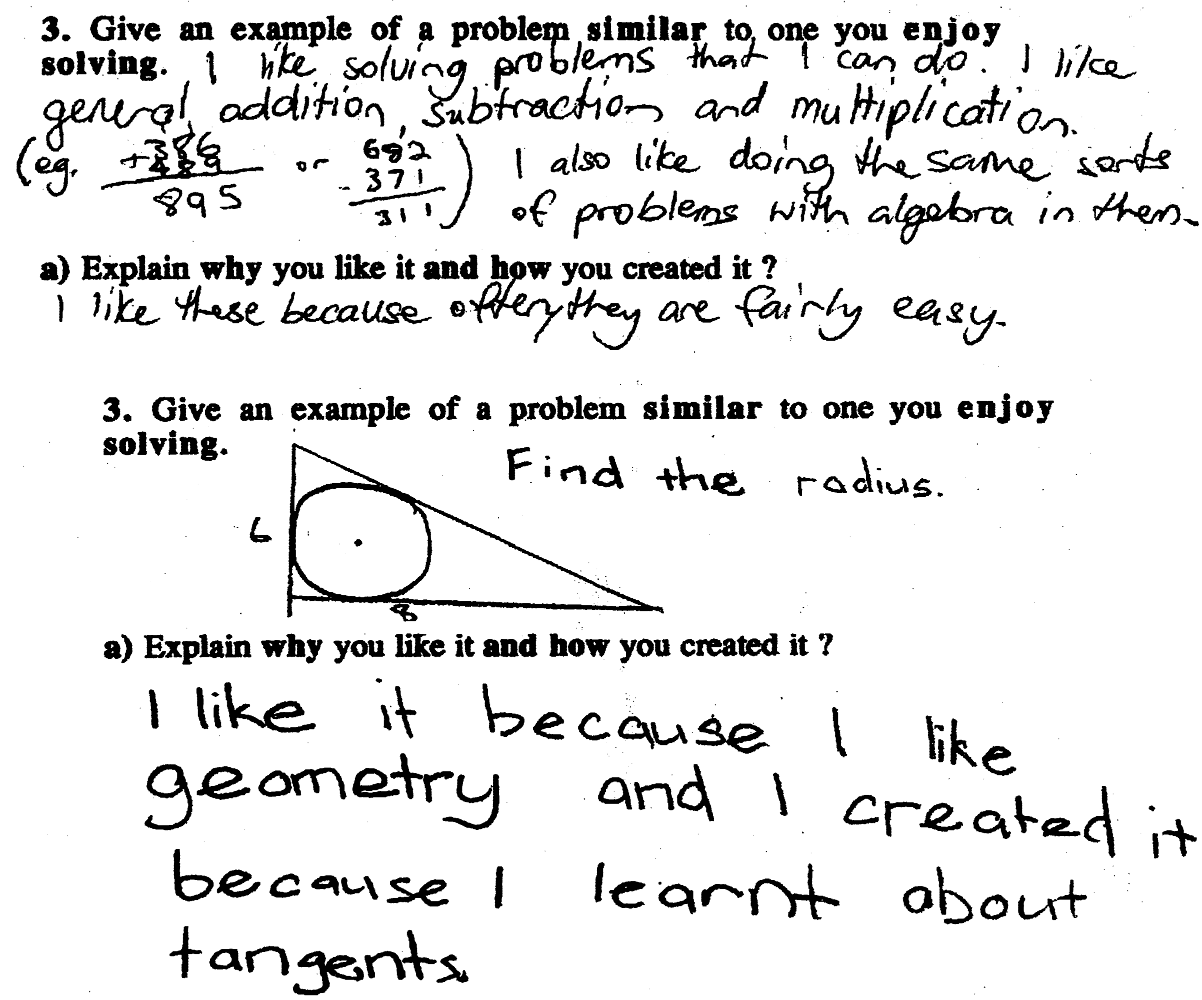


Figure 1. Mathematics problem posed by one student for a mathematics competition.



3. Give an example of a problem similar to one you enjoy solving. I like solving problems that I can do. I like general addition, subtraction and multiplication. (eg. 386 + [illegible] = 895 or 682 − 371 = 311) I also like doing the same sorts of problems with algebra in them.

a) Explain why you like it and how you created it ? I like these because often they are fairly easy.

3. Give an example of a problem similar to one you enjoy solving. Find the radius.

a) Explain why you like it and how you created it ? I like it because I like geometry and I created it because I learnt about tangents.

Figure 2. Mathematics problems created by two students to illustrate problems they enjoy.

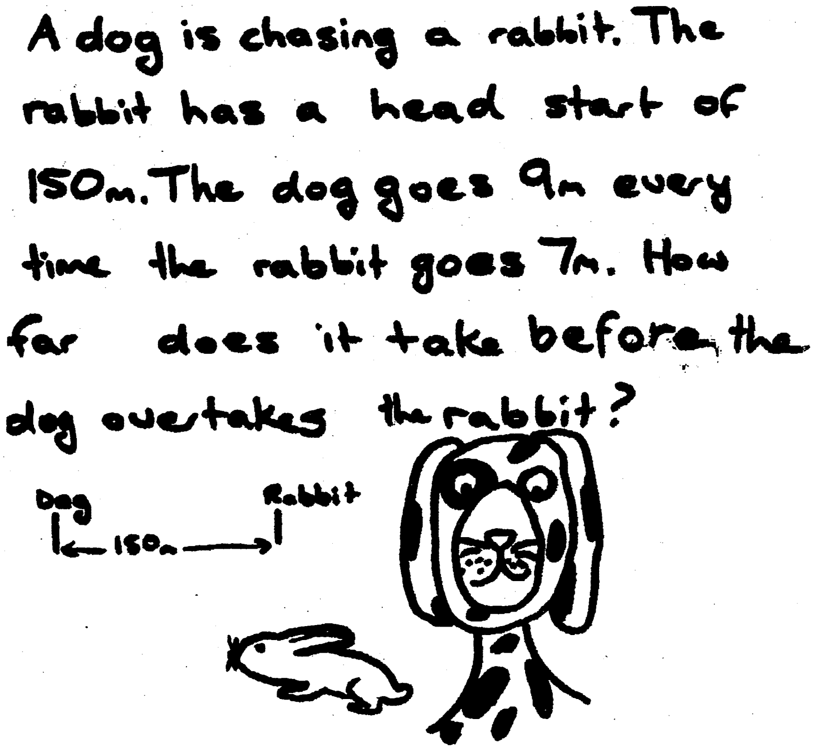
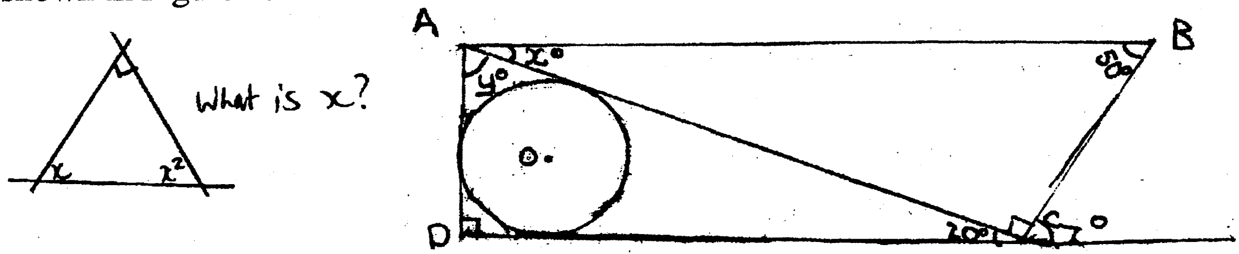


Figure 3. Problem posed by a student when asked to create a question that she would find difficult to solve.

Semi-Structured Problem-Posing Situations

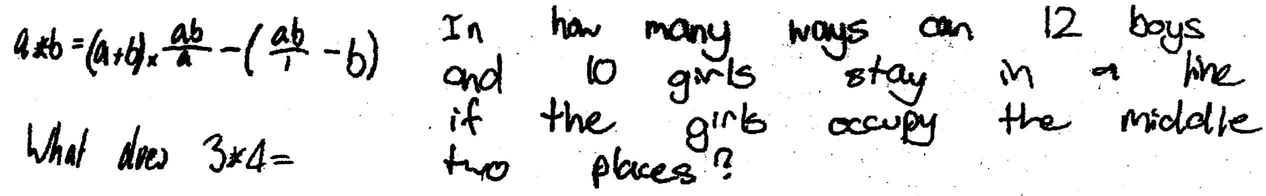
The semi-structured problem-posing situations used in the study ranged from situations incorporating unfinished structures to posing sequences of interconnected problems. We will mention only a few here. For example, students were asked to pose a problem which involved the use of a concept of the right-angled triangle—two responses are shown in Figure 4.



(a) (b)

Figure 4. Two problems posed by different students involving right-angled triangles.

Responses to another form of semi-structured problem posing are shown in Figure 5. These illustrate the posing of a class of problems related to a specific solution method—such as the use of the Pigeon-hole Principle, permutations, combinations, and working backwards.



(a) (b)

Figure 5. Two problems posed by different students which involve (a) constructing a new operation by working backwards, and (b) permutations.

Semi-structured problem-posing situations can also involve giving the students unfinished problem structures, and asking them to describe what kind of problems could be created on the basis of the information given. The unfinished problem structures can be given either by a picture, equation, calculation or inequality. Figure 6 gives a student's response to the question "Make up as many problems as you can using the following calculation: 3 x 25 + 15 ÷ 5 – 4."

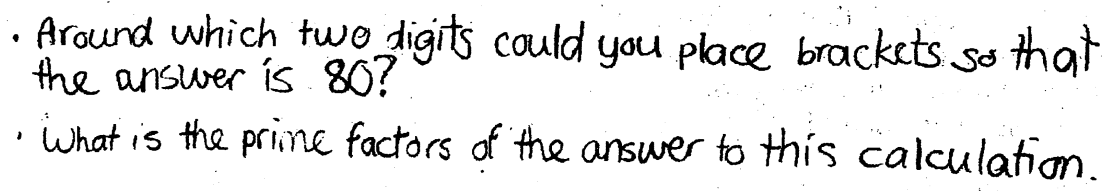


Figure 6. Problem posed in response to an unfinished problem structure.

Asking students to restate a problem when its solution is given, or presenting students with a part of the problem statement and a set of possible answers was another form of semi-structured problem-posing situation which was given to the students.

Structured Problem-Posing Situations

Three categories of structured problem-posing situations, based on a specific problem, were trialed. They were aimed at helping students to understand particular problems and solution structures, and to explore possible interrelationships between problem statement and solution ideas.

The section of a problem in which the question was asked was omitted, and students were asked to pose a series of possible questions and to put them in a suitable order. In other cases they were asked to add structure and to pose a question (see, for example,

Figure 7) or to find the surplus information and to improve the problem structure. The following situation provided a starting point for the student's response in Figure 7:

> Last night there was a party and the host's doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that, three more guests arrived than had arrived on previous ring.
> Ask as many questions as you can. Try to put them in a suitable order.

A) Every 5th person is a child and every 2nd child brings a dog.
B) There is a room in the house especialy for dogs.
C) the room can house 15 dogs how many times does the doorbell ring if the room is full of dogs. And if the first time 1 child arrives and brings his dog and if 4 more people arrive each doorbell ring than last doorbellring.

Figure 7. A posed problem similar to a given problem, but with added structure.

During, before and after solving a specific problem students were asked, on a regular basis, to suggest changes in the problem which might (or which do not) affect the solution method. Students were also asked to suggest a problem which resembled a given problem but might have a different solution method, and to pose a problem which is the inverse of the given problem

Restating a problem on the basis of its solution was another problem-posing situation used in the study. Students were asked to pose problems with different task formats, including "regular" problems, and multiple choice questions. In some cases the problem statement was given by a series of pictures. Improving the characteristics of a written solution by determining the main steps in the given solution and improving the language was another of the problem-posing activities in which the class engaged.

Conclusion

Although problem posing has had greater attention and recognition in recent years, the lack of a research framework which links problem posing, problem solving and school curricula has reduced the credibility of research in this area, and has delayed any systematic implementation of problem-posing situations into mathematics classrooms. This paper, with its three categories of problem-posing situations, and its set of problem-posing examples in each of these categories, is a step towards bridging the gap.

Acknowledgements: We wish to express our sincere thanks to all students who participated in the study.

References

Australian Education Council. (1991). *A national statement on mathematics for Australian schools. Canberra: Author.*

Brown, S., & Walter, M. I. (Eds.). (1993). *Problem Posing: Reflections and applications*. Hillsdale, NJ: Lawrence Erlbaum.

Brown, S., & Walter, M. I. (1983; 1990). *The art of problem posing*. Philadelphia, P. A: Franklin Institute Press.

Bush, W., & Fiala, A. (1986). Problem Stories: New Twist on Problem Posing. *The Arithmetic Teacher, 34*(4), 6-9.

Dillon, J. T. (1982). Problem finding and solving, *Journal of Creative Behaviour, 16*, 97-111.

Duncker, K. (1945). On problem solving. *Psychological Monographs, 58* (5, Whole No. 270).

Education Department of Western Australia. (1994). *Mathematics student outcome statements with pointers and work samples.*

Einstein, A., & Infeld, L. (1938). *The evolution of physics* (p. 92). New York: Simon and Schuster.

Ellerton, N. F. (1986a). Children's made-up mathematical problems: A new perspective on talented mathematicians. *Educational Studies in Mathematics, 17,* 261-271.
Ellerton, N. F. (1986b). Mathematics problems written by children. *Research in Mathematics Education in Australia* (December), 32-44.
Ellerton, N. F. (1988). Exploring children's perception of mathematics through letters and problems written by children. In A. Borbas (Ed.), *Proceedings of the Twelfth International Conference for the Psychology of Mathematics Education* (Vol.1, pp. 280-287). Veszprem (Hungary): International Group for the Psychology of Mathematics Education.
Hart, K. (Ed.). (1981). *Children's understanding of mathematics: 11-16.* London: John Murray.
Hashimotto, Y. (1987). Classroom parctice of problem solving in Japanese elementary schools. In Becker, J. P. & Miwa, T. (Eds), *Proceedings of the U. S.-Japan seminar on mathematical problem solving* (pp. 94-119). Carbon dale, IL: Southern Illinois University.
Kennedy, W. (1985). Writing letters to learn math. *Learning,* (Feb.), 59-60.
Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123-147). Hillsdale, NJ; Erlbaum.
Krutetskii, V. A., (1976). *The psychology of mathematics abilities in schoolchildren.* Chicago: University of Chicago Press.
Mamona-Downs, J. (1993).On anaysing problem posing. In I. Hirabayashi, N. Nohda, K. Shigematsu & F. L. Lin (Eds.), *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education* (Vol. III, pp. 41-47). Tsukuba (Japan): International Group for the Psychology in Mathematics Education.
National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics. Reston,* VA: Author.
Resnick, L. B., & Klopher, L. E., (Eds.) (1989). *Toward thinking curriculum: Current cognitive research.* Alexander, VA: Association for Supervision and Curriculum Development.
Richardson, J., & Williamson, P. (1982). Towards autonomy in infant mathematics. *Research in Mathematics Education in Australia,* 109-136.
Shukkwan, S. L. (1993). Mathematical problem posing: The influence of task formats, mathematics knowledge, and creative thinking. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. L. Lin (Eds.), *Proceedings of the 17th International Conference for the Psychology of Mathematics Education* (Vol III. pp. 33-40). Tsukuba (Japan): International Group for the Psychology in Mathematics Education.
Silver, E. (1993). On mathematical problem posing. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. L. Lin (Eds.). *Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 66-85). Tsukuba (Japan): International Group for the Psychology in Mathematics Education.
Silver, E. A. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. *International Reviews on Mathematical Education, 2,* 67-72.
Silver, E. A., & Cai, J. (1993). Mathematical problem posing and problem solving by middle school students. In C. A. Maher, G. A. Goldin & R. B. Davis (Eds.), *Proceedings of PMENA 11,* New Brunswick NJ: Vol. 1. pp. 263-269). Rutgers University.
Silver, E. A., & Mamona, J. (1989). Problem posing by middle school teachers. In C. A. Maher, G. A. Goldin & R. B. Davis (Eds.), *Proceedings of the eleventh annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 263-269). New Brunswick, NJ: PME-NA.
Silver, E., A., Kilpatrick, J., & Schlesinger, B., (1990). *Thinking through mathematics: Fostering inquiry and communication in mathematics classrooms.* (Eds. D. Wolf D. & R. Orril), College Entrance Examination Board, New York.
Smilansky, J. (1984). Problem solving in the quality of invention. *Journal of Educational Psychology,* 76, pp. 377-386.
Stacey, K. (1995). The challenges of keeping open problem-solving open in school mathematics. *International Reviews on Mathematical Education, 2,* 62-67.
Stover, G. B. (1982). *Structural variables affecting mathematical word problem difficulty in 6th-graders.* Unpublished doctoral dissertation, University of San Francisco. (*Dissertation Abstracts International, 42,* 5050A)
Van den Brink, J. (1987). Children as arithmetic book authors. *For the Learning of Mathematics, 7,* 44-48.
Winograd, K. (1991). *Writing, solving and sharing original math story problems: Case studies of fifth grade children's cognitive behaviour.* Unpublished doctoral thesis, University of Northern Colorado.
Writz, R. W. & Kahn, E. (1982). Another look at application in elementary school mathematics. The *Arithmetic Teacher, 30,* 21-25.
Zuckerman, H. (1977). *The scientific elite,* New York: Free Press.